Rigorous Proof of Pseudospin Ferromagnetism in Two-Component Bosonic Systems with Component-Independent Interactions

Kun Yang[1,2] and You-Quan Li[2]
[1]National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32306
[2]Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, P. R. China
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For a two-component bosonic system, the components can be mapped onto a pseudo-spin degree of freedom with spin quantum number $S = 1/2$. We provide a rigorous proof that for a wide-range of real Hamiltonians with component independent mass and interaction, the ground state is a ferromagnetic state with pseudospin fully polarized. The spin-wave excitations are studied and found to have quadratic dispersion relations at long wave length.

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There has been an explosion of interest in the physics of interacting bosonic systems since the realization of Bose-Einstein condensation in trapped alkali gases [1]. Recently, much attention has focused on bosons with internal degrees of freedom. These discrete internal degrees of freedom come from the (nearly) degenerate hyperfine levels or other sources that give rise to multiple components for the bosons. Indeed, recent studies [2,3,4,5,6,7,8] demonstrated that the multi-component bosonic systems have much richer physics than their single-component counterpart due to the additional degrees of freedom.

Despite intensive research, there has been relatively few rigorous results on these interesting systems. In this paper we study an example of such systems, namely a two-component bosonic system with component-independent mass and interactions [9]. Obviously the two components of the boson can be mapped to a pseudo-spin degree of freedom with spin quantum number $S = 1/2$ (*i.e.*, pseudo-spin "up" state representing one component, pseudo-spin "down" state representing the other component), and the interaction is pseudo-spin independent, thus the system is invariant under pseudo-spin $SU(2)$ rotation. We prove rigorously that for a large class of real Hamiltonians, the ground state of the system is a fully polarized pseudo-spin ferromagnet. The origin of the ferromagnetism is neither spin-dependent interaction (as there is none in our model), nor interparticle interaction as Coulomb interaction driven ferromagnetism in electronic systems; instead it is due to the fact the kinetic energy term of a bosonic system forbids the ground state wave function to change sign, thus forcing a totally symmetric spatial wave function and hence a totally symmetric (and fully polarized) pseudospin wave function. Thus the ferromagnetism is driven by *kinetic* energy in our system. We will also study the ferromagnetic spin wave spectra of the system under various conditions. For simplicity we will refer to the pseudospin of the bosons as spin from now on.

Consider the following Hamiltonian describing N identical bosons:

$$H = \sum_i \left[\frac{\mathbf{p}_i^2}{2m} + U(\mathbf{r}_i)\right] + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j). \qquad (1)$$

Here U is the trapping potential and V is the two-particle interaction. Our proof can also be generalized to cases with multi-body (three and above) interactions when present. Since H is spin independent, we have $[H, \mathbf{S}_i] = 0$, where $\mathbf{S}_i$ is the spin operator of ith particle. In particular, it possesses the global $SU(2)$ symmetry: $[H, \mathbf{S}_{tot}] = 0$, with $\mathbf{S}_{tot} = \sum_i \mathbf{S}_i$. Thus the eigenkets of H may be chosen to be simultaneous eigenkets of $\mathbf{S}_{tot}^2$ and $S_{tot}^z = (N_\uparrow - N_\downarrow)/2$.

We now proceed by considering the ground state of an *enlarged* Hilbert space for H, namely that of N *distinguishable* particles, *i.e.*, no permutation symmetry is required for the wave function. Since H is spin independent and commutes with S_i^z for every $1 \leq i \leq N$, in this *enlarged* Hilbert space we can always choose the eigenkets of H to be simultaneous eigenkets of S_i^z, so that the wave functions take a factorized form [10]:

$$\psi(\mathbf{r}_1, \sigma_1, \cdots, \mathbf{r}_N, \sigma_N) = \phi(\mathbf{r}_1, \cdots, \mathbf{r}_N)\chi(\sigma_1, \cdots, \sigma_N). \qquad (2)$$

Since the eigen energy is independent of the spin wave function, we only need to focus on the spatial wave function of the ground state: $\phi_0(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)$. Since H is real, ϕ_0 can be chosen to be real as well. We now prove that ϕ_0 can be chosen to be real and non-negative, using a well-known trick. If ϕ_0 contains both positive and negative parts, we can then construct a trial wave function that is non-negative: $\tilde{\phi}_0(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N) = |\phi_0(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N)|$, and show that $\langle\tilde{\phi}_0|H|\tilde{\phi}_0\rangle = \langle\phi_0|H|\phi_0\rangle$. It is easy to see that the expectation values of the potential terms are the same; the expectation values of the kinetic energy term are also the same because the integrand is the exactly the same everywhere $\phi_0 \neq 0$, while the singularity of $\nabla^2\tilde{\phi}_0$ where $\phi_0 = 0$ is not strong enough to overcome the speed that $\tilde{\phi}_0 \to 0$ and makes no contribution. Thus we find from variational principle that the ground state energy $E_0 \leq \langle\tilde{\phi}_0|H|\tilde{\phi}_0\rangle$. The equal sign holds only when $\tilde{\phi}_0$ happens to be the true ground state, which is *not* the case under generic

situations. Thus generically the ground state wave function ϕ_0 is non-negative definite, and moreover, positive definite for non-singular U and V [11]. This also implies that the ground state must be *non-degenerate*, because if there existed two different ground state wave functions that are both non-negative, it is always possible to construct new ground state wave functions that have both positive and negative parts by making linear combination of the two.

Having established that ϕ_0 is non-degenerate and non-negative definite under generic situations, it follows that it must also be totally symmetric: $P_{ij}\phi_0 = \phi_0$, where P_{ij} is the permutation operator for particles i and j. This is because $[P_{ij}, H] = 0$ thus a *non-degenerate* eigenket of H must also be an eigenket of P_{ij}, and the fact ϕ_0 is non-negative ensures the eigenvalue can only be $+1$.

We can now prove that the ground state wave function of the spin-1/2 *bosonic* system must take the form:

$$\psi_0^B(\mathbf{r}_1, \sigma_1, \cdots, \mathbf{r}_N, \sigma_N) = \phi_0(\mathbf{r}_1, \cdots, \mathbf{r}_N)\chi_{N/2}(\sigma_1, \cdots \sigma_N), \tag{3}$$

where $\chi_{N/2}(\sigma_1, \cdots \sigma_N)$ is a spin wave function with quantum number $S_{tot} = N/2$. First of all, ψ_0^B is symmetric under permutation, since both ϕ_0 and $\chi_{N/2}$ are symmetric under permutation. Secondly, ψ_0^B is an eigen wave function of H with eigen value E_0, thus it must be one of the ground states. Lastly but most importantly, we need to prove the lowest energy state with other $S_{tot} = S < N/2$ quantum numbers have energies higher than E_0. Without losing generality we can choose to focus on states with $S^z_{tot} = S_{tot} = S$. Such a state takes the following form:

$$\begin{aligned}\psi_S^B(\mathbf{r}_1, \sigma_1, \cdots, \mathbf{r}_N, \sigma_N) &= \sum_P \phi_S(\mathbf{r}_{P(1)}, \cdots, \mathbf{r}_{P(N)}) \\ &\times |\uparrow_{P(1)} \cdots \uparrow_{P(N_1)}; \downarrow_{P(N_1+1)} \cdots \downarrow_{P(N)}\rangle, \end{aligned} \tag{4}$$

where P stands for permutation, and $N_1 = N/2 + S$ is the number of up spin particles. In order for ψ_S^B to be an eigen wave function of H with eigen value E, ϕ_S must be an eigen spatial wave function of H with eigen value E, which can be chosen to be real. Now we prove $E > E_0$. Since ψ_S^B is a common eigen ket of $\mathbf{S}^2_{tot}$ and S^z_{tot} with $S^z_{tot} = S_{tot}$, we have

$$S^+_{tot}|\psi_S^B\rangle = \sum_i S_i^+|\psi_S^B\rangle = 0. \tag{5}$$

Since S^+_{tot} does *not* annihilate any term of $|\psi_S^B\rangle$ expanded as in eq. (4), and matrix elements of S_i^+ are real and positive in the S^z representation, in order for eq. (5) to be valid ϕ_S must contain both positive and negative parts. Thus $E > E_0$. We have thus proved that the ground state of the bosonic system described by the Hamiltonian (1) must have $S_{tot} = N/2$ with degeneracy $2S_{tot} + 1 = N + 1$; if there exists an *infinitesimal* magnetic field, the degeneracy will be lifted and the spins will be fully-polarized along the direction of the field.

Using similar methods one can also prove the same is true for lattice Hamiltonians of the form:

$$H = -\sum_{(ij),\sigma} t_{ij}(b^\dagger_{i\sigma}b_{j\sigma} + b^\dagger_{j\sigma}b_{i\sigma}) + \sum_i U_i n_i + \sum_{(ij)} V_{ij} n_i n_j, \tag{6}$$

where $b_{i\sigma}$ is the boson annihilation operator for site i and spin component σ, t_{ij} are real and *positive* hopping matrix elements, V_{ij} is the two-body interaction potential, and $n_i = b^\dagger_{i\uparrow}b_{i\uparrow} + b^\dagger_{i\downarrow}b_{i\downarrow}$ is the total boson number on site i. The lattice structure is not important and hence not specified.

We emphasize our proof does *not* assume the presence of BEC; in fact it is valid in the absence of BEC as well, and we will discuss an example of this situation later. On the other hand it is crucial that the Hamiltonian H is *real*, so that the eigen wave functions can be chosen to be real; the proof is no longer valid when the Hamiltonian H is complex, for example when the bosons are charged and moving in a magnetic field (with *no* Zeeman coupling so that the SU(2) symmetry is present), or the bosons are placed in a rotating trap.

A physical consequence of the ferromagnetism of the ground state is the presence of ferromagnetic spin wave excitations. In the following we study the spin wave spectra of two different situations, one in the continuum and the other in a lattice model.

Consider a continuum system described by the Hamiltonian Eq. (1) with $U = 0$, so that the system is translationally invariant and all eigenkets of H can be chosen to have a momentum quantum number $\mathbf{k}$. A natural variational wave function for a ferromagnetic spin-wave state takes the form

$$|\mathbf{k}\rangle = S^-_{\mathbf{k}}|0\rangle, \tag{7}$$

where $|0\rangle$ is the ground state with $S_{tot} = S^z_{tot} = N/2$, and

$$S^-_{\mathbf{k}} = (1/\sqrt{N})\sum_j e^{i\mathbf{k}\cdot\mathbf{r}_j}S^-_j. \tag{8}$$

This is, of course, a straightforward generalization of the single-mode-approximation (SMA) for *spinless* bosons [12]. Thus a variational approximation (or upper bound) for the spin-wave spectrum is

$$E(k) = f(k)/s(k) = \frac{\hbar^2 k^2}{2m}, \tag{9}$$

where

$$f(k) = \langle\mathbf{k}|(H - E_0)|\mathbf{k}\rangle = \frac{\hbar^2 k^2}{2m} \tag{10}$$

is the same oscillator strength as in SMA, while here

$$s(k) = \langle \mathbf{k}|\mathbf{k}\rangle = \langle 0|S^{+}_{-\mathbf{k}}S^{-}_{\mathbf{k}}|0\rangle = 1 \tag{11}$$

is the *spin* structure factor. In the SMA for spinless bosons the collective mode spectrum is linear for small k because the *density* structure factor is linear in k for small k [12]; here the spin-wave spectrum is quadratic as expected for Heisenberg ferromagnets, and we find that it is bounded from above by the single particle spectrum.

For bosons moving in a periodic potential [13], the system can be appropriately described by a tight-binding lattice Hamiltonian of the form (6). In particular, if the periodic potential is strong enough the system loses BEC and becomes a Mott insulator [13]. In this case the charge degree of freedom of the bosons are frozen, but the spin degree of freedom are still active and as we show below, they can be described by the Heisenberg model with *ferromagnetic* couplings. For simplicity we consider a boson Hubbard model with nearest neighbor hopping and on-site repulsion, on a simple cubic lattice:

$$\begin{aligned} H &= -t\sum_{<ij>,\sigma}(b^{\dagger}_{i\sigma}b_{j\sigma} + b^{\dagger}_{j\sigma}b_{i\sigma}) + \sum_i (U/2)(n_i-1)^2 \\ &= -t\sum_{<ij>,\sigma}(b^{\dagger}_{i\sigma}b_{j\sigma} + b^{\dagger}_{j\sigma}b_{i\sigma}) - (U/2)\sum_i n_i \\ &+ \sum_i U n_i(n_i-1)/2 + const., \end{aligned} \tag{12}$$

and assume the number of bosons is the same as the number of lattice sites, thus there is one boson on every site in average. For very large U/t the bosons can not hop from one site to another due to the large potential energy cost, and the system is in the Mott insulator phase with *no* BEC. A straightforward second-order perturbation calculation in t/U yields an effective Heisenberg spin Hamiltonian for the system:

$$H = J\sum_{<ij>}(\mathbf{S}_i\cdot\mathbf{S}_j + 3/4) \tag{13}$$

with *ferromagnetic* coupling:

$$J = -4t^2/U. \tag{14}$$

And the spin-wave spectrum takes the *exact* form

$$E(\mathbf{k}) = (8t^2/U)(3-\cos k_x - \cos k_y - \cos k_z) \tag{15}$$

in this limit. This is exactly the opposite for the Hubbard model at half-filling for electrons, where the large U/t limit leads to a Heisenberg *antiferromagnet*. We note in passing that in *electronic* systems, ferromagnetism is usually associated with *itinerant* electrons while localized electrons usually give rise to antiferromagnetism; thus two-component bosons trapped to lattice potentials provide new opportunities to study lattice ferromagnets.

We now turn our discussion to the relation between our work and existing theoretical work on this and related fields.

Ho and Yin [3] considered the general spin structure of Bose-Einstein condensates of bosons with arbitrary spin. In their work they assumed weakly interacting bosons thus to a very good approximation, all bosons occupy the lowest-energy orbital state. They pointed out for spin-1/2 bosons due to Bose statistics, the bosons must also occupy the same spin state and the system is a ferromagnet, and very appropriately termed the ferromagnetic state "statistical ferromagnet". Our work has considerable overlap with theirs in spirit, and certainly comes to the same conclusion as theirs. On the other hand our proof is rigorous and more general; it applies to interacting bosons with arbitrary interaction strength and in particular, strongly interacting bosonic systems without BEC. We have thus generalized the statistical ferromagnet to a much wider class of bosonic systems.

Rojo [6] studied two-component bosonic systems with *component-dependent* interactions, in which the intercomponent interaction is always more repulsive or less attractive than the intracomponent interaction. He showed that in this case the ground state is "fully polarized" in the sense that only bosons of one component appear. In the spin analogy the interaction he considered has Ising anisotropy, and the system is an Ising ferromagnet. Thus his work is complementary to ours, which studies a Heisenberg ferromagnet. By the same token, in his model if the intercomponent interaction were less repulsive or more attractive than the intracomponent interaction, the interaction would have XY anisotropy and the system would be an XY ferromagnet. He further found that as the two interactions become closer and closer to each other (or one is approaching the Heisenberg point), the velocity of one of the two linear collective modes vanishes. This, as we see here, is because at the Heisenberg point the spin wave spectrum has *quadratic* dispersion at long wave length.

Li and coworkers [14] have studied one-dimensional SU(2) bosons with δ-function interactions. They used Bethe ansatz to obtain the ground and low-lying state spectra and quantum numbers, and showed the ground state is fully spin polarized and there is a quadratic spin wave mode. Their results certainly agree with the general conclusions of the present work.

Pu *et al.* [4] studied possibility of ferromagnetic states in bosonic systems originated from dipole-dipole interactions. In their model the ferromagnetism comes from *interactions* that are magnetic in the first place, which is very different from our case, where the ferromagnetism is essentially *kinetic energy* driven.

Thus far, our discussions have been focusing on bosonic systems. Fermionic systems are of course very different, due to the difference in statistics. On the other hand in two-dimensions (2D) the statistics of the particles can

be transformed through flux attachment transformations [15]. Thus our results also have implications on electrons in a strong magnetic field in the quantum Hall regime. For example, it gives a new perspective to the spin/pseudospin ferromagnetism for single/bi-layer quantum Hall systems at Landau level filling factor at $\nu = 1$ [16,17], as the electrons can be mapped to bosons carrying one flux quantum, and the flux carried by the bosons *cancel* that of the external magnetic field *in average*, thus at the mean-field level these systems are mapped onto boson systems at *zero* magnetic field, precisely the system we study here! Using the same transformation one can also map spin-1/2 bosons in a magnetic field with $\nu = 1$ to fermions with zero field, where one expects a *singlet* ground state. This has indeed been found to be the case in finite-size studies, for Coulomb interaction [18].

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